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TITLE: INDICES IN CUBIC NUMBER FIELDS AND THUE EQUATIONS

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Let $f \in \mathbb{Z}[X,Y]$ be a homogeneous irreducible polynomial of degree $n \geq 3$ and k be an integer. In 1909 Thue proved that the diophantine equation

$$f(x,y) = k$$

has only finitely many solutions in integers x and y.
His method does not give explicitly the solutions. In this talk, first we review old and new results on this direction.

We investigate the theory of *indices in cubic number fields* . We then obtain new method to study Thue Equations. We obtain precise results when f is irreducible binary cubic form.
As a consequence of our study we will be able to obtain information on the number of integers and also of the number of rational points of the *Mordell's elliptic curves* :

$$E_d : dy^2 = x^3 + Ax + B.$$

These families are extremely studied in the literature and are sources of intensive research at present in the field of elliptic curves.
In particular, there is :
• Connection between the number of integers points on the curves E_d and the rank of E_d.
• Connection with Birch-Swinnerton-dyer conjecture and special values at integers of the L-function associated to E_d.

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