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Methods for Numerical Construction of s-Orthogonal Polynomials

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Abstract: Let $\mathrm{d}\lambda(t)$ be a positive measure on $\mathbb{R}$ with finite or unbounded support, for which all moments $\mu_k = \int_{\mathbb{R}} t^k \,\mathrm{d}\lambda(t)$, $k \geq 0$, exist and are finite, and $\mu_0 > 0$. For each $n, s \in \mathbb{N}$, it is well-known that the general Gauss-Turán quadrature formula

$$\int_{\mathbb{R}} f(t)\mathrm{d}\lambda(t) = \sum_{\nu=0}^{2s}\sum_{k=1}^{n} A_{i,k} f^{(\nu)}(\tau_\nu) + R_{n,2s}(f)$$

is exact for all polynomials of degree not exceeding $2(s+1)n-1$, i.e., $R_{n,2s}(f) = 0$ for all $f \in \mathcal{P}_{2(s+1)n-1}$, where $\mathcal{P}_m$ is the set of all algebraic polynomials of degree at most m. The nodes τ_ν, $\nu = 1,2,\dots,n$, are the zeros of the monic polynomial $\pi_{n,s}(t)$, which minimizes the integral $F(a_0, a_1, \dots, a_{n-1}) = \int_{\mathbb{R}} [\pi_{n,s}(t)]^{2s+2}\,\mathrm{d}\lambda(t)$, where $\pi_{n,s}(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$. This minimization gives the conditions

$$\frac{1}{2s+2}\frac{\partial F}{\partial a_k} = \int_{\mathbb{R}} [\pi_{n,s}(t)]^{2s+1} t^k \,\mathrm{d}\lambda(t) = 0, \qquad k = 0,1,\dots,n-1,$$

and such polynomials $\pi_{n,s}(t)$ are known as *s-orthogonal polynomials* on $\mathbb{R}$ with respect to the measure $\mathrm{d}\lambda(t)$. For $s = 0$ they reduce to the standard orthogonal polynomials. The first attempt in the numerical construction of s-orthogonal polynomials with respect to an even weight function on a symmetric interval $(-c, c)$ was given in 1986 by Vincenti [6], who constructed an iterative process for computing their coefficients. He applied it to Legendre s-orthogonal polynomials, but the corresponding numerical results for coefficients were obtained only for low degrees of polynomials n and relatively small values of s, because the process becomes numerically unstable when n and s increase.

In 1987 we presented a stable method for numerically constructing s-orthogonal polynomials and their zeros, which opened the door to the numerical construction of not only s-orthogonal polynomials, but also to the quadrature rules of Gauss-Turán type and other generalized quadratures of Gauss-Stancu type (cf. [1]–[5]).

Using ideas on reninterpretation the orthogonalitiy conditions for s-orthogonal polynomials in terms of the ordinary orthogonality, in this lecture we present a method of continuation for a stable constructinion of s-orthogonal polynomials. In addition, the construction of parameters in the generalized Gauss-Turán quadrature formulas is also given. Examples for different classical measures are presented.

Keywords: Positive measure, Moments, Gauss-Turán quadrature formula, Orthogonal and s-orthogonal polynomials, Numerical construction, Method of continuation.

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