Two parametric kinds of Apostol type numbers and polynomials related to Eisenstein series and Dedekind sums

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Abstract: In this paper, we give a brief survey and history of generating functions for Apostol type numbers and polynomials, the Eisenstein series and the Dedekind sums. The aim of this paper is to give some relations and formulas about the special numbers and polynomials which related to trigonometric function. By using generating functions and their functional equations, we obtain various identities formulas and identities including two variables of Apostol-Bernoulli and two variables of Apostol-Genocchi polynomials. Moreover, we give further remarks and observations on relations between Apostol type numbers and polynomials, the Eisenstein series, the Dedekind sums and cotangent functions.

Keywords: Generating function, Functional equation, Trigonometric function, Two variables of Apostol-Bernoulli, Two variables of Apostol-Genocchi polynomials, Eisenstein series, Dedekind sums and cotangent functions.

Introduction

Throughout this paper, we use the following notations, definitions and relations. Here and in the following, let $\mathbb{N} = \{1,2,\ldots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ denote the set of integers and $\mathbb{C}$ denote the set of complex numbers.

In work of Srivastava et al. [4] defined the following two parametric kinds of Apostol-Bernoulli polynomials:

$$F_{BC}(t,x,y;\lambda) = \frac{t}{\lambda e^t - 1} e^{xt} \cos(yt) = \sum_{n=0}^{\infty} \mathfrak{B}_n^{(C)}(x,y;\lambda) \frac{t^n}{n!}, \tag{1}$$

$$F_{BS}(t,x,y;\lambda) = \frac{t}{\lambda e^t - 1} e^{xt} \sin(yt) = \sum_{n=0}^{\infty} \mathfrak{B}_n^{(S)}(x,y;\lambda) \frac{t^n}{n!} \tag{2}$$

(cf. [4]).

In [4], two parametric kinds of Apostol-Genocchi polynomials are defined by means of the following generating functions:

$$F_{GC}(t,x,y;\lambda)=\frac{2t}{\lambda e^t+1}e^{xt}\cos(yt)=\sum_{n=0}^{\infty}\mathcal{G}_n^{(C)}(x,y;\lambda)\frac{t^n}{n!}, \qquad (3)$$

$$F_{GS}(t,x,y;\lambda)=\frac{2t}{\lambda e^t+1}e^{xt}\sin(yt)=\sum_{n=0}^{\infty}\mathcal{G}_n^{(S)}(x,y;\lambda)\frac{t^n}{n!} \qquad (4)$$

(cf. [4]).

Identities and Relations

In this section, we give some identities, relations and formulas including two parametric kinds of Apostol-Genocchi polynomials and two parametric kinds of Apostol-Bernoulli polynomials.

Theorem 1. Let $n\in\mathbb{N}_0$. Then we have

$$\mathcal{G}_n^{(S)}(x,y+w;\lambda)=\sum_{j=0}^{\left[\frac{n}{2}\right]}(-1)^j\binom{n}{2j}\left(w^{2j}\mathcal{G}_{n-2j}^{(S)}(x,y;\lambda)+y^{2j}\mathcal{G}_{n-2j}^{(S)}(x,w;\lambda)\right).$$

Proof. Combining (4) with the following well-known identity

$$\sin(yt+wt)=\sin(yt)\cos(wt)+\cos(yt)\sin(wt),$$

we get

$$F_{GS}(t,x,y+w;\lambda)=F_{GS}(t,x,y;\lambda)\cos(wt)+F_{GS}(t,x,w;\lambda)\cos(yt).$$

By using above functional equation, we have

$$\sum_{n=0}^{\infty}\mathcal{G}_n^{(S)}(x,y+w;\lambda)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\mathcal{G}_n^{(S)}(x,y;\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty}(-1)^n\frac{(wt)^{2n}}{(2n)!}$$
$$+\sum_{n=0}^{\infty}\mathcal{G}_n^{(S)}(x,w;\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty}(-1)^n\frac{(yt)^{2n}}{(2n)!}.$$

Therefore

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(S)}(x, y+w; \lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{j=0}^{\left[\frac{n}{2}\right]} (-1)^j \binom{n}{2j} \left(w^{2j} \mathcal{G}_{n-2j}^{(S)}(x, y; \lambda) + y^{2j} \mathcal{G}_{n-2j}^{(S)}(x, w; \lambda) \right) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result.

Theorem 2. Let $n \in \mathbb{N}_0$. Then we have

$$\mathcal{G}_n^{(S)}(x, y-w; \lambda) = \sum_{j=0}^{\left[\frac{n}{2}\right]} (-1)^j \binom{n}{2j} \left(w^{2j} \mathcal{G}_{n-2j}^{(S)}(x, y; \lambda) - y^{2j} \mathcal{G}_{n-2j}^{(S)}(x, w; \lambda) \right).$$

Proof. Combining (4) with the following well-known identity

$$\sin(yt - wt) = \sin(yt)\cos(wt) - \cos(yt)\sin(wt),$$

we get

$$F_{GS}(t, x, y-w; \lambda) = F_{GS}(t, x, y; \lambda)\cos(wt) - F_{GS}(t, x, w; \lambda)\cos(yt).$$

From the above functional equation, we have

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(S)}(x, y-w; \lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \mathcal{G}_n^{(S)}(x, y; \lambda) \frac{t^n}{n!} \sum_{n=0}^{\infty} (-1)^n \frac{(wt)^{2n}}{(2n)!}$$
$$- \sum_{n=0}^{\infty} \mathcal{G}_n^{(S)}(x, w; \lambda) \frac{t^n}{n!} \sum_{n=0}^{\infty} (-1)^n \frac{(yt)^{2n}}{(2n)!}.$$

Therefore

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(S)}(x, y-w; \lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{j=0}^{\left[\frac{n}{2}\right]} (-1)^j \binom{n}{2j} \left(w^{2j} \mathcal{G}_{n-2j}^{(S)}(x, y; \lambda) - y^{2j} \mathcal{G}_{n-2j}^{(S)}(x, w; \lambda) \right) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result.

Conclusions

Using generating functions for two variable Apostol type polynomials with their functional equations are we investigate and sutudy many properties of these polynomials. These polynomials related to not only trigonometric functions, but also the Eisenstein series and the Dedekind sums. We also give recent advences and studies on these polynomials and the Eisenstein series, the Dedekind sums and cotangent functions. The results of this paper will be potentially used by resercers who are study on the content of that of paper.

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